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The Magnus expansion for classical Hamiltonian systems

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Abstract. Lie operator techniques and the Magnus expansion are developed in the framework of classical mechanics. This leads to an exponential perturbation theory that preserves the canonical character at each order of approximation. The treatment is kept as close as possible to the quantum mechanical case in order to take full advantage of the properties of the expansion. The explicit relationship with secular perturbation theory is established and a recursive procedure for obtaining higher-order approximants is provided. Finally, the formalism is applied to two problems of physical interest.

1. Introduction

There are many areas of physics where perturbation techniques of classical mechanics are of great usefulness: optics [1], celestial mechanics [2], problems concerning non-integrable systems [3] or control of beam dynamics in accelerators [4], to quote just a few examples. Analytic description of the evolution of physical systems in phase space is rarely possible so that in most cases one has to handle approximate solutions. Various methods have been devised over the years to find the best approximants for each situation. Among them Lie algebraic methods proved to be much superior to earlier perturbation techniques and led to powerful computational algorithms [4–10].

Besides their practical interest, Lie techniques also provide a deep connection between quantum and classical mechanics. For instance, in quantum mechanics time-displacement can be described by a unitary operator acting on states of the Hilbert space. In turn, the evolution in classical mechanics can be considered as a symplectic map (i.e. a canonical transformation) in phase space. This canonical transformation may be thought of as being performed by a Lie operator, which is the key point of this work. Our goal is to apply the so-called Magnus expansion [11–13] to classical mechanics in analogy with what has been done in a variety of time-dependent quantum mechanical problems [14]. The Magnus expansion enables one to construct exponential representations of the time-displacement operator. For this reason, it is also called exponential perturbation theory. In the present paper we elaborate the appropriate formalism to solve classical Hamiltonian systems via an exponential-like canonical transformation.

We carry out a treatment of the time evolution of classical systems as close as possible to the quantum mechanical formalism. This permits to use efficiently some well established properties of the quantum Magnus expansion. In particular, we show

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that the formalism preserves the symplectic character of the mapping, unlike the secular perturbation scheme. This feature is the dual of the unitarity property of the time-evolution operator of a quantum system.

In section 2 we set up the notation and present some concepts concerning Lie operators. In section 3 we introduce the classical Magnus formalism, study some of its properties and establish the connection with secular perturbation theory. In section 4 we explain the recursive methods for obtaining higher-order Magnus approximants. An application of the preceding techniques is carried out for two examples of physical interest in section 5. Finally in section 6 we discuss the results.

2. Lie operators

In this section we introduce briefly the notation and some mathematical tools that will be required in the following. More complete discussions can be found in the specific literature [2, 4, 5, 6, 8, 9].

Let H be a Hamiltonian with generalized coordinates q_i, and generalized momenta p_i, in a $2N$-dimensional phase space. For convenience, we introduce the vector ξ whose $2N$ components are $(\xi_1, \ldots, \xi_{2N}) = (q_1, \ldots, q_N, p_1, \ldots, p_N)$. Using the notation of [8], the Lie operator associated with a function f on the phase space will be represented by $:f:$, and its action on any function g defined by

$$:f:g = [f, g] \tag{2.1}$$

where the square bracket stands for the Poisson bracket of classical mechanics

$$[f, g] = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} = \sum_{i,j=1}^{2N} \frac{\partial f}{\partial \xi_i} J_{ij} \frac{\partial g}{\partial \xi_j}. \tag{2.2}$$

Here J is the symplectic $2N \times 2N$ matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

On another hand, we denote the commutator of two Lie operators by $\{:f:, :g:\}$. It turns out that the Lie operator of a Poisson bracket of two functions equals the commutator of their two associated Lie operators

$$:[f, g]: = \{:f:, :g:\}. \tag{2.3}$$

For a phase space of dimension $2N$, the Lie operator associated to the function f (sometimes referred to as *Lie derivative*, or *vector field* of the observable f) is given by

$$:f: = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} = \sum_{i,j=1}^{2N} \frac{\partial f}{\partial \xi_i} J_{ij} \frac{\partial}{\partial \xi_j}. \tag{2.4}$$

We are going to deal with operator-like functions of the form $\exp(:f:)$, which are defined by their Taylor series expansion. Thus, we introduce a Poisson nested bracket and an iterated commutator. Whenever a power of $:f:$ appears within brackets

we must interpret it as a repeated action. Specifically, we build up nested Poisson brackets of the form

$$:f:^0 g = g, \qquad :f:^1 g = [f,g] \qquad \cdots \qquad :f:^n g = \underbrace{[f,[f,\ldots,[f}_{n \text{ times}},g]\ldots]] \tag{2.5}$$

$$\exp(:f:)g = \sum_{n=0}^{\infty} \frac{1}{n!} :f:^n g = g + [f,g] + [f,[f,g]] + \cdots .$$

For the sake of illustration, we write down the Baker–Hausdorff identity [11,12] in this notation:

$$e^{:f:} :g: e^{-:f:} = :e^{:f:} g:. \tag{2.6}$$

Eventually, since we will be involved with time derivatives of exponential Lie operators we pave the road by writing down the following useful formula due to Wilcox [12]:

$$\frac{\partial}{\partial t} e^{:f(t):} = e^{:f:} \int_0^1 dx\, :e^{-x:f:} \dot{f}: = e^{:f:} : \frac{1 - e^{-:f:}}{:f:} \dot{f} : . \tag{2.7}$$

The last form is obtained by using (2.5), (2.6) and the overdot indicates time differentiation.

3. The classical Magnus expansion

Consider an explicitly time-dependent Hamiltonian $H(\xi, t)$. The trajectories $\xi(t)$ in the phase space, starting from initial conditions $\xi(t_0)$, can be interpreted as the result of a, in general nonlinear, symplectic map

$$\xi(t) = \mathcal{M}(t)\xi(t_0). \tag{3.1}$$

It can be shown that $\mathcal{M}$ obeys the equation of motion [6, 8]

$$\dot{\mathcal{M}} = -\mathcal{M} : H : \tag{3.2}$$

with the initial condition $\mathcal{M}(t_0) = I$, in order to guarantee that $\mathcal{M}$ is continuously connected with the identity. When the condition $[H(t), H(t')] = 0$ holds for all t and t', then the solution $\exp(-\int_{t_0}^{t} dt' :H(t'):)$ follows for equation (3.2). Otherwise, a formal solution for a general H is given by $\mathcal{T} \exp(-\int_{t_0}^{t} dt' :H(t'):)$, where $\mathcal{T}$ stands for Dyson's chronological product.

When comparing time-dependent problems in classical and quantum physics a comment is in order: strictly speaking one can always avoid explicitly time-dependent problems in classical Hamiltonian mechanics. The price to be paid is just to increase the number of degrees of freedom and appropriately change the Hamiltonian. This, however, may complicate the algebraic structure of the problem.

The Magnus method consists in looking for a solution of the form $\mathcal{M}(t) = \exp(:\Omega(t):)$, for equation (3.2). This form has to be taken as an ansatz to solve equation (3.2) and it is not meant to imply that it exists for a general Hamiltonian.

Assuming it does, the question is then to find the differential equation for $:\Omega:$. From the Wilcox formula it follows that

$$\dot{\mathcal{M}} = \mathcal{M} : \frac{1 - e^{-:\Omega:}}{:\Omega:} \dot{\Omega} : . \tag{3.3}$$

Comparing this result with the equation of motion gives the relation

$$\frac{1 - e^{-:\Omega:}}{:\Omega:} \dot{\Omega} = -H. \tag{3.4}$$

Consequently, Ω obeys the equation of motion

$$\dot{\Omega} = \frac{-:\Omega:}{1 - e^{-:\Omega:}} H. \tag{3.5}$$

Setting now $:\Omega: = \sum_{i=1}^{\infty} :\Omega_i:$, we obtain the successive Magnus approximants, $:\Omega_i:$, by identifying terms of the same order in the characteristic coupling constant of the Hamiltonian. The first three terms are:

$$\begin{aligned} \Omega_1 &= -\int_{t_0}^{t} dt_1 H_1 \\ \Omega_2 &= \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 [H_1, H_2] \\ \Omega_3 &= -\frac{1}{6} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 ([H_1, [H_2, H_3]] + [[H_1, H_2], H_3]) \end{aligned} \tag{3.6}$$

with $H_i \equiv H(t_i)$.

When the Hamiltonian has a solvable piece, it may be advisable to integrate it explicitly in order to increase the rate of convergence of the expansion. In quantum mechanical calculations this is achieved by changing from the Schrödinger picture to the interaction picture [15]. Here, the formalism of Lie operators allows us to set up a similar procedure.

Suppose that the Hamiltonian decomposes as follows

$$H = H_0 + H_1 \tag{3.7}$$

where the time-displacement operator associated with H_0 has the exact solution

$$\exp(:\Omega_0(t, t_0):) \equiv \exp\left(-\int_{t_0}^{t} dt' :H_0(t'): \right).$$

Assuming solutions of the form $\mathcal{M} = \mathcal{M}_I \exp(:\Omega_0:)$, it turns out that $\mathcal{M}_I$ obeys the evolution equation

$$\dot{\mathcal{M}}_I = -\mathcal{M}_I :H_I: \tag{3.8}$$

with the transformed Hamiltonian

$$H_I = e^{:\Omega_0:} H_1. \tag{3.9}$$

Then, one seeks a solution $\mathcal{M}_I = \exp(:\Omega^I:)$, by means of the Magnus formalism.

So, equation (3.8) shows that the time evolution operator $\mathcal{M}_I$ evolves according to the 'perturbation' Hamiltonian H_I. This change into the interaction picture can also be accomplished within the more usual formalism. Let $\xi = \xi(t, \chi)$ be the solution of the dynamical problem associated with H_0, with initial conditions $\xi(0, \chi) = \chi$. If instead of taking χ as given initial values we interpret them as variable parameters then one can easily prove that $\xi = \xi(t, \chi)$ is a time-dependent canonical transformation and that $\chi(t)$ satisfies canonical equations of motion with Hamiltonian H_I.

In the quantum mechanical case the Magnus expansion in the Schrödinger picture may fail to converge owing to the presence of poles in the complex t-plane. We must point out that the divergencies are originated just by the diagonal piece of the Hamiltonian H in that picture. This problem, which has been widely discussed in the literature [16], also occurs for the Lie operators in classical mechanics. As a matter of fact there is no general rule as how to decompose H in a soluble piece and a perturbation. Sometimes the most obvious choice spoils the convergence of the Magnus expansion, as will be seen in section 5.

The only general result about convergence is a theorem due to Magnus [11] which states the existence of $:\Omega:$ whenever any pair of its eigenvalues λ_k satisfies $\lambda_k - \lambda_j \neq \pm 2\pi n\mathrm{i}$ (n integer). Otherwise, the derivative of the Magnus operator does not exist at these points. However, this result is not of much utility as the exact solution must be known in advance.

Eventually, the symplectic character of the Magnus expansion to all orders follows directly from the properties of Lie operators [5]. It is interesting to observe that in secular perturbation theory the canonical character of the transformation is lost when we truncate the expansion. In this respect the canonical perturbation theory (e.g. the Poincaré–Von Zeipel theory) behaves more like the Lie approach.

4. Recursive generation of the Magnus expansion terms

Although the first three terms of the Magnus series are explicitly given in equation (3.6), there is no recipe to build up the general term. However, there are a number of recursive algorithms that have been developed in the quantum mechanical case [17, 18]. In this section we proceed to fit them into our scheme, and give just their final form. An exhaustive proof can be found in [17].

The property expressed by equation (2.3) allows us to work either with functions or with operators. If one uses the operator formalism then the methods exposed in [17] apply without further ado. It is, however, more convenient to deal with Poisson brackets and, afterwards, to compute the corresponding Lie derivative. In this case the method of interest is the so-called commutator approach which provides the derivatives of successive terms in the expansion of the Magnus operator. The general formula reads

$$:\dot{\Omega}_1: = -:H: \qquad :\dot{\Omega}_n: = \sum_{k=1}^{n-1} \frac{B_k}{k!} :S_n^{(k)}: \qquad (n \geqslant 2) \tag{4.1}$$

where B_n are Bernoulli numbers ($B_1 = -1/2$) and $S_n^{(k)}$ can be obtained recursively

$$S_n^{(k)} = \sum_{m=1}^{n-k} [\Omega_m, S_{n-m}^{(k-1)}] \qquad (2 \leqslant k < n-1)$$
$$S_n^{(1)} = -[\Omega_{n-1}, H] \qquad S_n^{(n-1)} = -[\Omega_1^{n-1}, H]. \tag{4.2}$$

Alternative schemes connect secular perturbation theory with the Magnus expansion. For the sake of completeness we give here some formulae that will be employed in section 5. If the perturbation series is written as $\mathcal{M} = I + \sum_{n=1}^{\infty} P_n$, then one finds

$$\begin{aligned} P_1 &= :\Omega_1: \\ P_2 &= :\Omega_2: + \frac{1}{2!}:\Omega_1:^2 \\ P_3 &= :\Omega_3: + \frac{1}{2!}(:\Omega_1::\Omega_2: + :\Omega_2::\Omega_1:) + \frac{1}{3!}:\Omega_1:^3 \\ &\vdots \end{aligned} \tag{4.3}$$

where the exponents and products stand for successive action of operators. The corresponding recursive procedure is developed in [17]. Here, the absence of commutators prohibits a structure in terms of Poisson brackets. Notice that these simple expressions provide the explicit connection between secular and exponential perturbation theory.

5. Some examples

(i) Consider the Hamiltonian of a linearly forced one-dimensional harmonic oscillator

$$H = \tfrac{1}{2}(p^2 + q^2) + f(t)q \tag{5.1}$$

with the initial conditions $q(0) = q_0$, $p(0) = p_0$. Here $f(t)$ is an arbitrary function of time and, without loss of generality, we have set $t_0 = 0$. According to equation (2.4) the associated Lie operator reads

$$:H: = (q + f(t))\partial_p - p\partial_q \tag{5.2}$$

where ∂_q, ∂_p, represent partial derivative operators. According to the discussion following equation (3.7), it is convenient to work in the interaction picture. Using equation (3.9), where $:\Omega_0: = t(p\partial_q - q\partial_p)$; and the Baker–Hausdorff identity, (2.6), we readily obtain

$$:H_I: = f(t)(\cos t\partial_p - \sin t\partial_q) \tag{5.3}$$

which may be thought of as corresponding to the function

$$H_I = f(t)(q\cos t + p\sin t). \tag{5.4}$$

In this case only the first order gives a non-vanishing contribution to Ω^{I} leading to the exact solution

$$:\Omega^{\mathrm{I}}: = F_s\partial_q + F_c\partial_p \tag{5.5}$$

$$F_s(t) \equiv \int_0^t \mathrm{d}t' f(t')\sin t' \qquad F_c(t) \equiv -\int_0^t \mathrm{d}t' f(t')\cos t'.$$

The exact mapping $\mathcal{M} = \exp(:\Omega^{\mathrm{I}}:)\exp(:\Omega_0:)$ yields eventually

$$\begin{aligned}\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \mathcal{M}\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} &= \mathrm{e}^{:\Omega^{\mathrm{I}}:}\begin{pmatrix} q_0\cos t + p_0\sin t \\ -q_0\sin t + p_0\cos t \end{pmatrix} \\ &= \begin{pmatrix} (q_0 + F_s)\cos t + (p_0 + F_c)\sin t \\ (p_0 + F_c)\cos t - (q_0 + F_s)\sin t \end{pmatrix}.\end{aligned} \tag{5.6}$$

Of course, the same result can be obtained by direct integration of the Hamilton equations. The derivation given above is merely intended as a check of the new method presented here.

(ii) Another widely investigated system is the parametrically driven harmonic oscillator. The Hamiltonian for this situation can be written as

$$H = \tfrac{1}{2}(p^2 + q^2) + f(t)q^2. \tag{5.7}$$

The corresponding Lie operator is given by

$$:H: = [1 + 2f(t)]q\partial_p - p\partial_q. \tag{5.8}$$

If we proceed as before and compute the Lie operator of the Hamiltonian in the interaction picture we get $:H_{\mathrm{I}}: = f(t)\ [\sin(2t)\,(p\partial_p - q\partial_q) + 2\cos^2(t)\,q\partial_p - 2\sin^2(t)\,p\partial_q)$, which contains an undesirable diagonal piece: $p\partial_p - q\partial_q$. This illustrates the discussion in section 3 concerning the choice of appropriate picture. Therefore we use equation (5.8) to carry out the calculation. Unlike the preceding example, here we are faced with an expansion that does not end. We compute the first and second orders of both the Magnus expansion and secular perturbation theory.

Substitution of equation (5.8) into equation (3.6) yields

$$\begin{aligned}:\Omega_1: &= F(t)(p\partial_q + q\partial_p) + [F(t) - (t - t_0)](q\partial_p - p\partial_q) \\ :\Omega_2: &= G(t)(p\partial_p - q\partial_q)\end{aligned} \tag{5.9}$$

with the following definitions

$$F(t) \equiv -\int_{t_0}^t \mathrm{d}t' f(t') \qquad G(t) \equiv -\int_{t_0}^t \mathrm{d}t'[F(t') + (t' - t_0)f(t')]. \tag{5.10}$$

The secular perturbation operators, P_1 and P_2, are readily obtained from equation (4.3).

The three operators:

$$:a: \equiv :pq: = p\partial_p - q\partial_q$$
$$:b: \equiv \tfrac{1}{2}:q^2 - p^2: = q\partial_p + p\partial_q$$
$$:c: \equiv \tfrac{1}{2}:q^2 + p^2: = q\partial_p - p\partial_q$$

span a closed algebra. Therefore, an exponential-like mapping, if it exists, must be of the form $\exp[\alpha(t):a: + \beta(t):b: + \gamma(t):c:]$. Substitution into equation (3.8) then leads to a system of coupled nonlinear differential equations for the three functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$. This is a well established alternative procedure [12] but the new system is usually even more intricate than the starting equations of motion. For later use it is, however, of interest to know the symplectic map generated by the above linear combination. After some algebra we get

$$e^{\alpha:a:+\beta:b:+\gamma:c:}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} [\cosh\eta - (\alpha/\eta)\sinh\eta]q + [(\beta-\gamma)/\eta]p\sinh\eta \\ [\cosh\eta + (\alpha/\eta)\sinh\eta]p + [(\beta+\gamma)/\eta]q\sinh\eta \end{pmatrix} \tag{5.11}$$

where $\eta^2 = \alpha^2 + \beta^2 - \gamma^2$. It is straightforward to verify that this transformation is canonical as its Jacobian W satisfies the symplectic condition $W^{\mathrm{T}}JW = J$.

We now choose a particular form of $f(t)$ for which the exact solution is known and compare it with the Magnus approximation. One of the simplest possible choices is the double step function

$$f(t) = \begin{cases} \epsilon/2 & 0 \leqslant t \leqslant T \\ 0 & t < 0,\ t > T. \end{cases} \tag{5.12}$$

For $t > T$ the exact trajectories then read

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = R(1;t,T)R(\omega;T,0)R(1;0,t_0)\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \tag{5.13}$$

in terms of initial conditions $q_0 = q(t_0)$, $p_0 = p(t_0)$ with $t_0 < 0$ and $\omega = \sqrt{1+\epsilon}$. Here R is the 2×2 matrix:

$$R(r;s,t) = \begin{pmatrix} \cos r(s-t) & r^{-1}\sin r(s-t) \\ -r\sin r(s-t) & \cos r(s-t) \end{pmatrix}. \tag{5.14}$$

Writing $:\Omega_1: + :\Omega_2: = \alpha(t):a: + \beta(t):b: + \gamma(t):c:$, and taking into account equations (5.9), (5.10) one finds

$$\begin{aligned} \alpha(t) &= \begin{cases} t_0\Theta(t) & t \leqslant T \\ (t_0 + t - T)\epsilon T/2 & t > T \end{cases} \\ \beta(t) &= -\Theta(t) \\ \gamma(t) &= t_0 - t - \Theta(t) \end{aligned} \tag{5.15}$$

where we have defined the function

$$\Theta(t) = \begin{cases} 0 & t < 0 \\ \epsilon t/2 & 0 \leqslant t \leqslant T \\ \epsilon T/2 & T < t. \end{cases} \tag{5.16}$$

The approximate trajectories up to second order are obtained from

$$\begin{pmatrix} q^{(2)}(t) \\ p^{(2)}(t) \end{pmatrix} = e^{:\Omega_1:+:\Omega_2:} \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}. \tag{5.17}$$

where the superscript indicates the order of the Magnus approximation. After computing the integrals of equation (5.10) and with the aid of equation (5.11) we get

$$\begin{aligned} q^{(2)} &= \left(\cosh\eta - \alpha\frac{\sinh\eta}{\eta}\right) q_0 + (\beta-\gamma)\frac{\sinh\eta}{\eta}\, p_0 \\ p^{(2)} &= \left(\cosh\eta + \alpha\frac{\sinh\eta}{\eta}\right) p_0 + (\beta+\gamma)\frac{\sinh\eta}{\eta}\, q_0 \end{aligned} \tag{5.18}$$

with α, β and γ given by equations (5.15), (5.16).

The same computation in the secular perturbation scheme is readily carried out by taking into account equation (4.3). The corresponding approximate trajectories up to second order are given by

$$\begin{pmatrix} q_s^{(2)}(t) \\ p_s^{(2)}(t) \end{pmatrix} = (I + P_1 + P_2) \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \tag{5.19}$$

where

$$\begin{aligned} q_s^{(2)}(t) &= [1 + \tfrac{1}{2}(\beta^2 - \gamma^2) - \alpha] q_0 + (\beta-\gamma) p_0 \\ p_s^{(2)}(t) &= [1 + \tfrac{1}{2}(\beta^2 - \gamma^2) + \alpha] q_0 + (\beta+\gamma) p_0. \end{aligned} \tag{5.20}$$

In figure 1 we have plotted the exact trajectories, equation (5.13), and the approximations provided by equations (5.15)–(5.20) for initial conditions $q_0 = 0$, $p_0 = 1.5$, $t_0 = -1$. The exact solution is represented by the solid curve. The dot-dashed and dotted lines stand for the second order of the Magnus expansion, equation (5.18), and secular perturbation theory, equation (5.20), respectively. The first-order approximations are quite close to that of second order and therefore, to simplify the figures, we have not plotted them. In figures 1(*a*) and 1(*b*) we have set $T = 5$ and $T = 1$ respectively, and $\epsilon = 0.2$, $t_0 \leqslant t \leqslant 6$ for both of them. Comparison of these figures permits an understanding of the effect of increasing or decreasing T, with ϵ fixed, on the approximate trajectory. It turns out that the smaller is T, the better is the fit to the exact solution. This agrees with the well established fact that in quantum mechanics the Magnus expansion gives the best approximation in the sudden perturbation regime. In figure 1(*c*) we have set $T = 5$, $t_0 \leqslant t \leqslant 6$, and increased the perturbation: $\epsilon = 0.3$. Comparison with figure 1(*a*) indicates that the Magnus approach up to second order becomes worse with increasing ϵ. Eventually, in figure 1(*d*) we have represented the results for $\epsilon = 0.3$, $T = 1$, over a longer time-interval: $t_0 \leqslant t \leqslant 20$. As a consequence of the canonical character of the Magnus approximation no secular behaviour appears, unlike secular perturbation theory which diverges except for very short times in all cases.

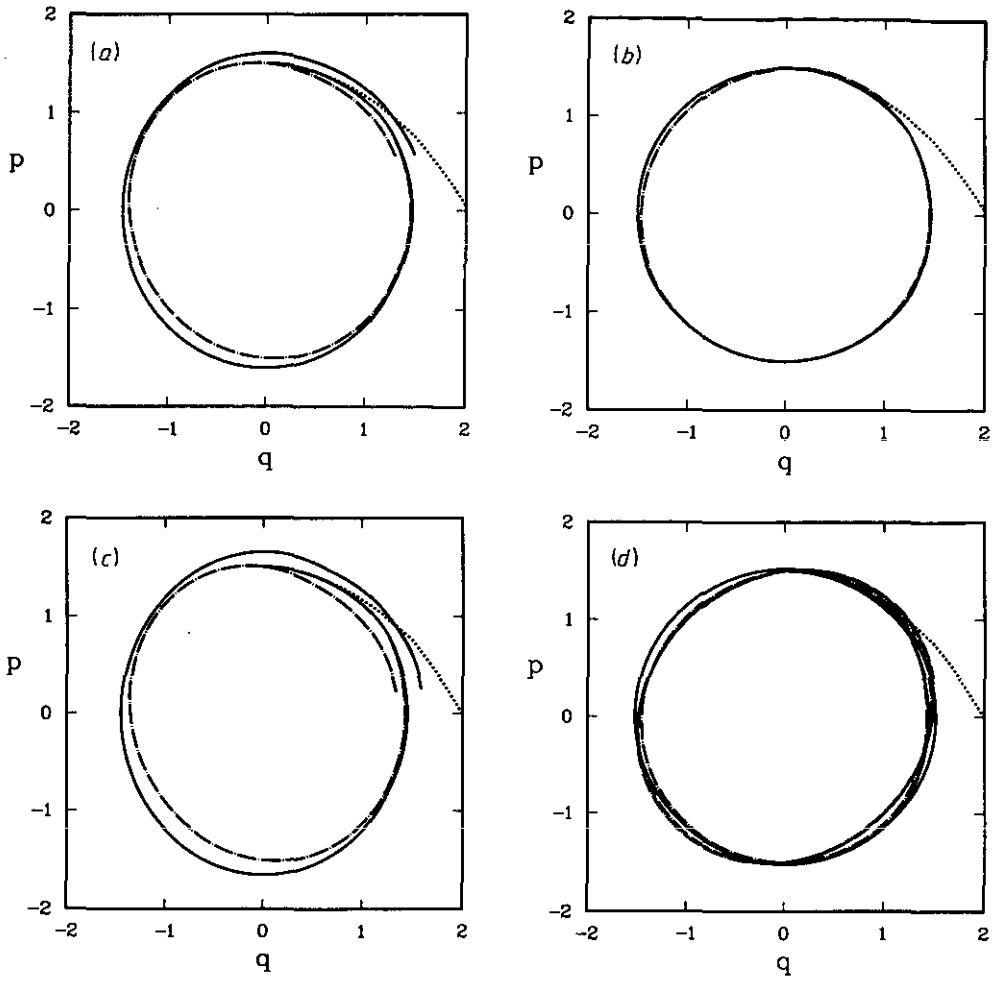

Figure 1. Trajectories in phase space generated by the Hamiltonian defined in equation (5.7) with input values: (*a*) $\epsilon = 0.2$, $T = 5$; (*b*) $\epsilon = 0.2, T = 1$; (*c*) $\epsilon = 0.3T = 5$; (*d*) $\epsilon = 0.3, T = 1$. The initial conditions are $q_0 = 0$, $p_0 = 1.5$, $t_0 = -1$, and the time-interval corresponds to (*a, b, c*) $t_0 \leqslant t \leqslant 6$ and (*d*) $t_0 \leqslant t \leqslant 20$. The solid line represents the exact solution. The dot-dashed curve stands for the second Magnus approximant according to equation (5.19). The dotted line corresponds to the second approximation of secular perturbation given by equation (5.21).

6. Discussion

We have adapted the Magnus formalism to classical Hamiltonian systems. The method does not require any generating function, yet produces a canonical transformation. The connection with secular perturbation theory has been explicitly shown too. We have also given some formulae and recursive procedures for computing the different orders of the expansion.

In analogy with quantum mechanics we have shown how to transform to an interaction picture in which part of the original Hamiltonian has been integrated out.

An application of the Magnus method to two physical examples has been carried

out. In the first one, the exact solution has been reached. In the second case an approximate solution up to second order has been obtained. For not too large values of the parameters it mimics very well the exact solution.

The perturbation approach which is closest to that of Magnus, in the sense that it uses Lie algebraic techniques, is that of Dragt and Finn [4,8] (DF). In their scheme, one deals with a Hamiltonian $H(\xi, t)$, for which the variables ξ are expressed with respect to some given trajectory. After that, one looks for factorized solutions

$$\mathcal{M} = \ldots \exp(:f_4:) \exp(:f_3:) \exp(:f_2:) \tag{6.1}$$

where each function f_n is a homogeneous polynomial of degree n in the variables $\xi(t_0)$. They give explicit formulae to compute up to $n = 6$. Formally at least, one could try to use the Baker–Campbell–Hausdorff (BCH) formula [11,12] to convert the above product of exponentials into a unique exponential. For this reason it is not surprising that the formulae of DF [8] for computing f_n have a structure similar to that of Magnus, equations (3.6), (4.1), (4.2). The convenience to deal with either a sole exponential or a product must be analysed in each situation. Here we restrict ourselves to point out that the construction of an exponential representation is possible by a direct but formal way. Of course the problem of convergence of the series remains open in the general situation.

We note in passing that the DF method could be adapted to work in the framework of quantum mechanics. Two different infinite product representations of the time evolution operator were formulated by Fer [19] and Wilcox [12]. The DF method provides a third alternative.

Other methods [3,7] have been devised for Hamiltonians which split into two solvable pieces, say $H = H_1 + H_2$, with solutions $\exp(-t:H_1:)$ and $\exp(-t:H_2:)$ respectively. Here we have taken $t_0 = 0$. The basic idea consists in supposing that an approximate symplectic map can be obtained in the form $\exp(-t:H_2:)\exp(-t:H_1:)$. Then, by using the BCH formula (truncated to some order) the product can be approximated by a unique exponential. To do this one proceeds by computing approximately the logarithm of the above expression. This recipe improves when one symmetrizes [7] with respect to one of the pieces of H. For instance, the approximant $\exp(-\frac{1}{2}t:H_2:)\exp(-t:H_1:)\exp(-\frac{1}{2}t:H_2:)$ seems to be better than the former one. More elaborate symmetrization procedures have also been proposed [7,20]. Notice, however, that this situation is more restrictive than the one we have treated here. As a matter of fact, one should compare $\exp(-t:H_2:)\exp(-t:H_1:)$ with $\exp(:\Omega^1:)\exp(-t:H_1:)$, where the interaction picture is defined by H_1. It is clear that the first product represents in general a poor substitute for the exact time-evolution operator.

The Magnus expansion and the reviewed methods are powerful tools for solving time-dependent problems. It should, however, be kept in mind that great care must be exercised in practical applications because convergence problems may arise.

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